Lecture 5

6. SIMPLEST HARMONIC CURRENT CIRCUITS

6.1. The Harmonic Current Circuit with a Series Connection of *R*, *L*, *C* Elements

Consider the networks in Fig 1.9, *a*. Let the voltage u of the source vary according to the harmonic law [8–10]. Its image in complex form can be written as

$$u = U_m \cos(\omega t + \Psi_U) = \dot{U}_m = U_m e^{j\Psi_U}.$$

According to Kirchhoff's voltage law we get

$$u = u_r + u_L + u_C$$

or

$$U_m \cos(\omega t + \psi_U) = ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i\,d\tau. \qquad (6.1)$$

The integral-differential equation (6.1) is the equation of electrical equilibrium for the circuit in Fig. 1.9, *a*. Let the current be:

$$i = I_m \cos(\omega t + \psi_i) = \dot{I}_m = I_m e^{j\psi_i}.$$

Write down the image of the equation (6.1) in complex form

$$\dot{U}_m = rI_m + j\omega L\dot{I}_m + \frac{\dot{I}_m}{j\omega C} = \left(r + j\omega L + \frac{1}{j\omega C}\right)\dot{I}_m = Z\dot{I}_m, \qquad (6.2)$$

where

$Z = r + j\omega L + \dfrac{1}{j\omega C} = r + j\left(\omega L - \dfrac{1}{\omega C}\right) = r + jx = Ze^{j\varphi}$ — complex impedance of a circuit;

$x = \omega L - \dfrac{1}{\omega C}$ — reactive impedance of a circuit;

$Z = \sqrt{r^2 + x^2} = \sqrt{r^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}$ — impedance of a circuit;

$\varphi = \arctan\dfrac{x}{r} = \arctan\dfrac{\omega L - \frac{1}{\omega C}}{r}$ — phase angle of a circuit — phase shift between the current and the voltage in a circuit.

Now from (6.2) we get

$$\dot{I}_m = \frac{\dot{U}_m}{Z} = \frac{U_m e^{j\cdot\Psi_u}}{Ze^{j\varphi}} = \frac{U_m}{Z}e^{j\cdot\Psi_u - \varphi} = I_m e^{j\cdot\Psi_i}. \qquad (6.3)$$

That is $I_m = \dfrac{U_m}{Z}$, $\Psi_i = \Psi_u - \varphi$, $\varphi = \Psi_u - \Psi_i$.

The voltage across the resistance R:

$$\dot{U}_{mr} = r\dot{I}_m = rI_m e^{j\cdot\Psi_i} = r\frac{U_m}{Z}e^{j(\Psi_u - \varphi)}. \qquad (6.4)$$

That is, the voltage across the active resistance r, according to (6.3) and (6.4), is in phase with the current and lags in phase from the voltage applied to the circuit by the angle φ.

The voltage across the inductance L:

$$\dot U_{mL} = j\omega L \dot I = \omega L I_m e^{j\Psi_i} e^{j\frac{\pi}{2}} = \omega L \frac{U_m}{Z} e^{j\left(\Psi_u-\varphi+\frac{\pi}{2}\right)}. \qquad (6.5)$$

That is, the voltage across the inductor L, according to (6.3) and (6.5), leads the current in phase by the angle $\frac{\pi}{2}$.

The voltage across the capacitance C:

$$\dot U_{mC} = \frac{I_m}{\omega \cdot C} e^{j\Psi_i} e^{-j\frac{\pi}{2}} = \frac{U_m}{\omega C Z} e^{j\left(\Psi_u-\varphi-\frac{\pi}{2}\right)}. \qquad (6.6)$$

I.e. voltage across the capacitance C, according to (6.3) and (6.6), lags in phase from the current by the angle $\frac{\pi}{2}$.

In going from the complex image to the original, we will obtain from (6.3)–(6.6):

$$i = \frac{U_m}{Z}\cos(\omega t + \Psi_u - \varphi),$$

$$u_r = \frac{rU_m}{Z}\cos(\omega t + \Psi_u - \varphi),$$

$$u_L = \frac{\omega L U_m}{Z}\cos\left(\omega t + \Psi_u - \varphi + \frac{\pi}{2}\right),$$

$$u_C = \frac{U_m}{\omega C Z}\cos\left(\omega t + \Psi_u - \varphi - \frac{\pi}{2}\right).$$

In Fig. 6.1 vector diagrams for the r, L, C — circuit (Fig. 1.9, *a*) are shown.

Here, in Fig. 6.1, *a*, the voltage $\dot U_m$ leads the current $\dot I_m$ in phase. The angle φ measured from the current to the voltage is positive. The circuit as a whole is inductive in nature.

In Fig. 6.1, *b* the voltage $\dot U_m$ lags from the current $\dot I_m$. The angle φ is negative. The circuit as a whole is capacitive in nature.

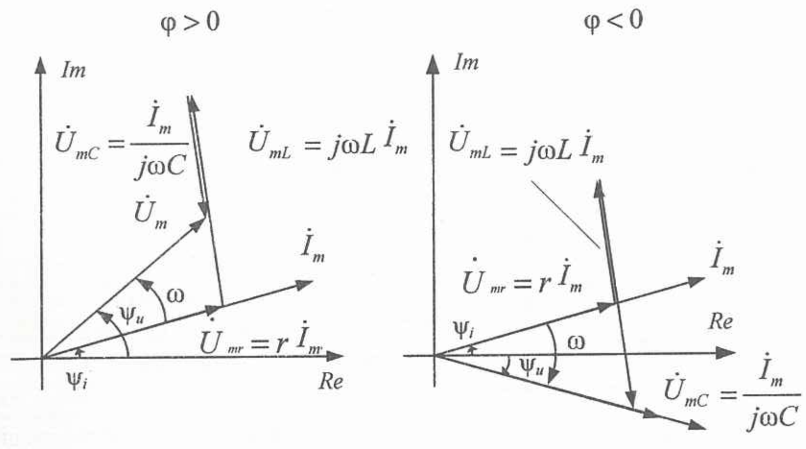


Fig. 6.1

Dividing all values of the vector diagrams in Fig. 6.1 by the current $\dot I_m$, we obtain the corresponding vector diagrams for resistances (Fig. 6.2).

Here the angle φ is measured from the active resistance r to the complex impedance Z. The vector diagrams in Fig. 6.2, *a* and *b* for an inductive load $(\varphi > 0)$ are equivalent. Also, the vector diagrams in Fig. 6.2, *c* and *d* for a capacitive load $(\varphi < 0)$ are equivalent. The triangles 0AB in Fig. 6.2 are resistance triangles.

Inductive reactance $x_L = \omega L$ and capacitive reactance $x_C = 1/\omega C$ depend on the frequency ω.

Dependence diagrams of the resistance r, inductive reactance x_L, capacitive reactance x_C, reactive reactance $x = \omega - 1/\omega C$ and impedance Z are depicted in Fig. 6.3.

It is obvious that at the frequency ω_0 we get:

$$x = x_L - x_C = \omega_0 L - \frac{1}{\omega_0 C} = 0,$$

$$\varphi = \arctan\frac{x}{r} = \arctan\frac{\omega_0 L - \dfrac{1}{\omega_0 C}}{r},$$

$$Z = r,\ \dot{I}_m = \frac{\dot{U}_m}{r},\ \dot{U}_{mL} = \dot{U}_{mC}.$$

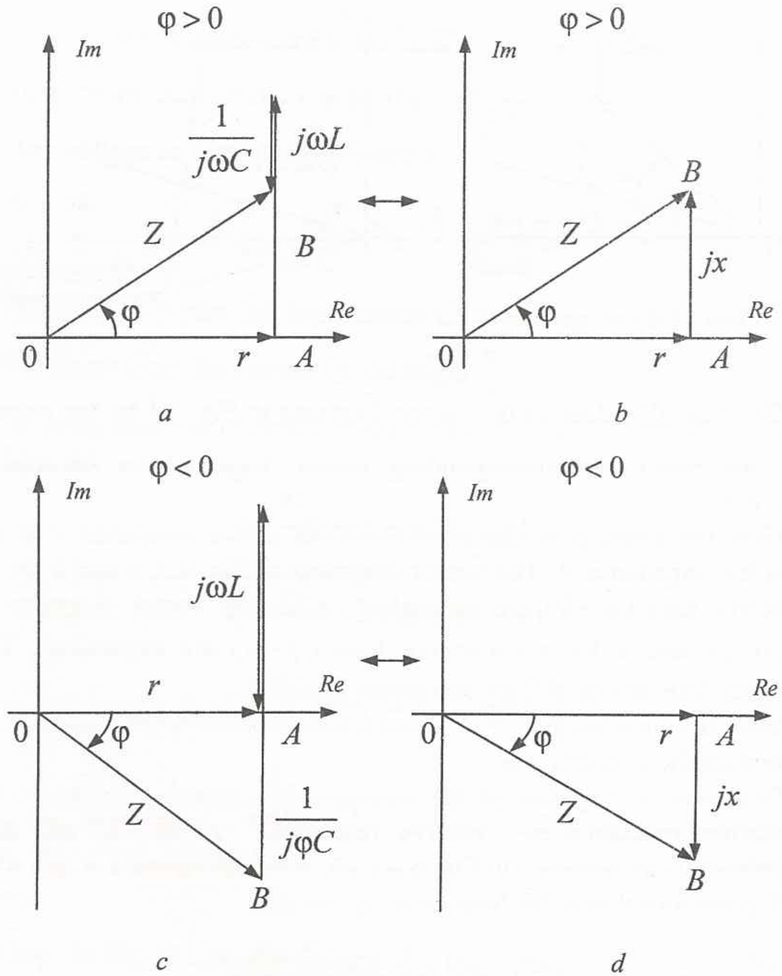


Fig. 6.2

It is obvious that at the frequency ω_0 we get:

$$x = x_L - x_C = \omega_0 L - \frac{1}{\omega_0 C} = 0,$$

$$\varphi = \arctan\frac{x}{r} = \arctan\frac{\omega_0 L - \dfrac{1}{\omega_0 C}}{r},$$

$$Z = r,\ \dot{I}_m = \frac{\dot{U}_m}{r},\ \dot{U}_{mL} = \dot{U}_{mC}.$$

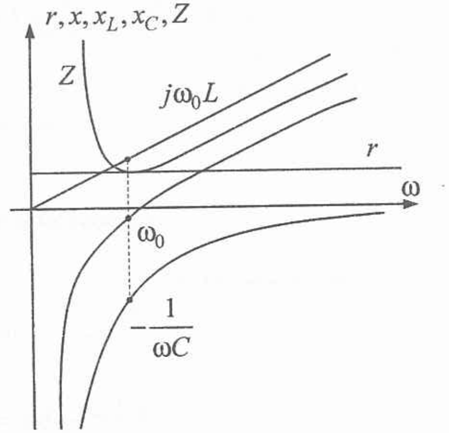


Fig. 6.3

This mode is referred to as voltage resonance and will be discussed in detail below.

Vector diagrams for the frequency ω_0 are shown in Fig. 6.4, *a* (for currents and voltages) and in Fig. 6.4, *b* (for resistances).

Reactive power at voltage resonance is equal to zero. Power in the circuit is purely active.

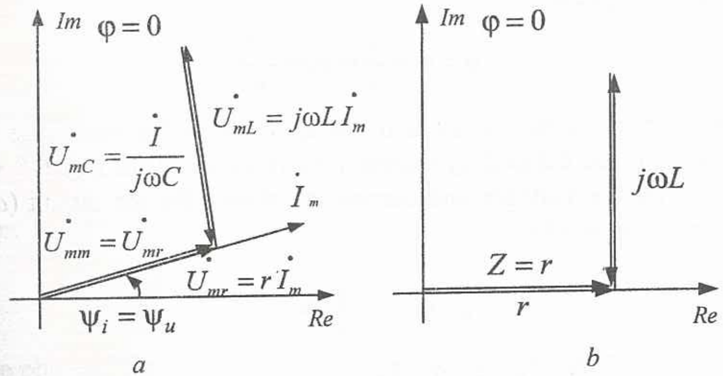


Fig. 6.4

6.2. The Harmonic Current Circuit with a Series Connection of R, L Elements

Relationships for this kind of circuit can be derived from the expressions of section 6.1 with $C \to \infty$. Indeed we get

$$x_C = \frac{1}{\omega C}.$$

That is, the capacitance C in Fig. 1.9, a can be replaced by a short-circuited jumper.

Then, we get

$$u = u_r + u_L$$

or

$$U_m \cos(\omega t + \Psi_u) = ri + L\frac{di}{d\tau}.$$

And we can write:

$$\dot{U}_m = (r + j\omega L)\dot{I}_m = Z\dot{I}_m,$$

where

$$Z = r + j\omega L = r + jx = Ze^{j\varphi}.$$

Reactance

$$x = x_L = \omega L.$$

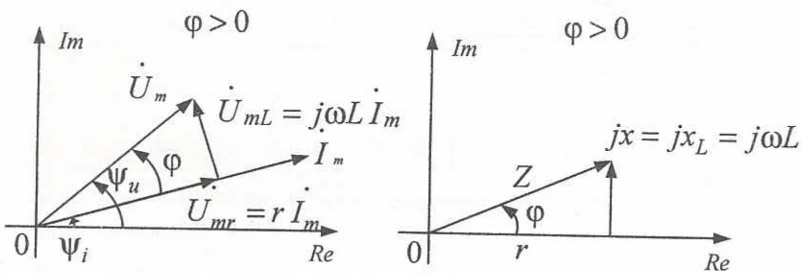
Impedance

$$Z = \sqrt{r^2 + x^2} = \sqrt{r^2 + (\omega L)^2}.$$

The phase angle

$$\varphi = \operatorname{atan}\frac{x}{r} = \operatorname{atan}\frac{\omega L}{r}.$$

The current in the circuit and voltages across the resistance r and inductance L are defined by expressions (6.3)–(6.5). In Fig. 6.5 vector diagrams for the voltages and current are shown for the circuit (a) and for the resistances (b).



a *b*

Fig. 6.5

6.3. The Harmonic Current Circuit with a Series Connection of *R*, *C* Elements

Relationships for this kind of circuit can be derived from expressions of section 6.1 when $L = 0$. Indeed we get

$$x_L = \omega L = 0.$$

That is the inductance L in Fig. 1.9, a can be replaced by a short-circuited jumper. Then we get

$$u = u_r + u_C$$

or

$$U_m \cos(\omega t + \Psi_u) = ri + \frac{1}{C} \pm \int_0^t i d\tau.$$

We can write from (6.2)

$$\dot{U}_m = \left(r + \frac{1}{j\omega C}\right)\dot{I}_m = Z\dot{I}_m,$$

where

$$Z = r + \frac{1}{j\omega C} = r - j\frac{1}{\omega C} = r + jx = Ze^{j\varphi}.$$

Reactance

$$x = x - x_C = -\frac{1}{\omega C}.$$

Impedance

$$Z = \sqrt{r^2 + x^2} = \sqrt{r^2 + \left(\frac{1}{\omega C}\right)^2}.$$

The phase angle

$$\varphi = \operatorname{atan}\frac{x}{r} = \operatorname{aran}\left(-\frac{1}{\omega Cr}\right) = -\operatorname{atan}\frac{1}{\omega Cr}.$$

The current in the circuit and voltages across the resistance R and capacitance C are defined by expressions (6.3)–(6.4) and (6.6).

In Fig. 6.6 vector diagrams for the voltages and current are presents for the circuit (a) and resistances (b).

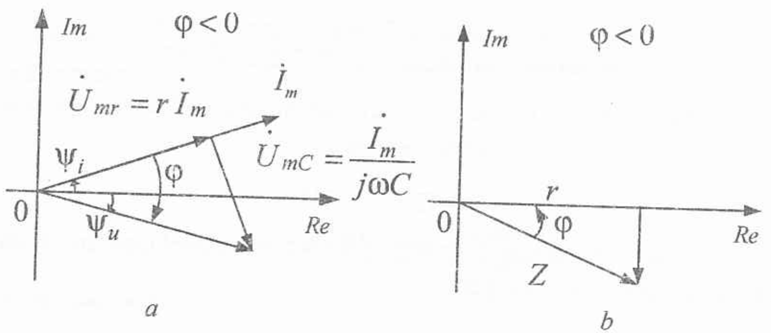


Fig. 6.6

6.4. The Harmonic Current Circuit with a Parallel Connection of *R, L, C* Elements

Consider the networks in Fig. 1.9, *b*, which is dual to the network in Fig. 1.9, *a*. It is obvious that all relationships for this network can be obtained from the expressions of sections 6.1–6.3 by means of dual substitution. Let the energy source create a current

$$j = I_m\cos(\omega t + \Psi_j) \doteq \dot I_m = I_m e^{j\Psi_j}.$$

According to Kirchhoff's current law we get

$$j = i_g + i_C + i_L$$

or

$$\dot J_m \cos(\omega t + \Psi_j) = ug + C\frac{du}{dt} + \frac{1}{L}\int_0^t u\,d\tau. \qquad (6.7)$$

The image of (6.7) in complex form can be written as

$$\dot J_m = g\dot U_m + j\omega C\dot U_m + \frac{\dot U_m}{j\omega L} = \left(g + j\omega C + \frac{1}{j\omega L}\right)\dot U_m = Y\dot U_m, \quad (6.8)$$

where

$Y = g + j\omega C + \frac{1}{j\omega L} = g - j\left(\frac{1}{\omega L} - \omega C\right) = g - jb = ye^{-j\varphi}$ — complex admittance of the circuit;

$b = \frac{1}{\omega L} - \omega C$ — susceptance of the circuit;

$y = \sqrt{g^2 + b^2} = \sqrt{g^2 + \left(\frac{1}{\omega L} - \omega C\right)^2}$ — admittance of the circuit;

$\varphi = \arctan\frac{b}{g} = \arctan\frac{\frac{1}{\omega L} - \omega C}{g}$ — phase angle of the circuit — phase shift between the current and the voltage in the circuit.

Now from (6.8) we get

$$\dot U_m = \frac{\dot J_m}{Y} = \frac{J_m e^{j\Psi_j}}{ye^{-j\varphi}} = \frac{J_m}{y}e^{j(\Psi_j + \varphi)} = U_m e^{j\Psi_u}. \qquad (6.9)$$

That is

$$U_m = \frac{J_m}{y};\ \psi_u = \psi_j + \varphi;\ \varphi = \psi_u - \psi_j.$$

The current in the conductance *g* is

$$\dot I_{mg} = g\dot U_m = gU_m e^{j\psi_u}. \qquad (6.10)$$

That is current in the active conductance *g*, according to (6.9) and (6.10) is in phase with the voltage and lags in phase from the source current by the angle φ.

The current in the capacitance *C* is

$$\dot I_{mC} = j\omega C\dot U_m = \omega CU_m e^{j\psi_u} e^{j\frac{\pi}{2}} = \omega CU_m e^{j\left(\psi_u + \frac{\pi}{2}\right)}. \qquad (6.11)$$

That is current in the capacitance *C*, according to (6.9) and (6.11), leads the voltage in phase by the angle φ.

The current in the inductance *L* is

$$\dot I_{mL} = \frac{\dot U_m}{j\omega L} = \frac{U_m e^{j\psi_u}}{\omega L}e^{-j\frac{\pi}{2}} = \frac{U_m}{\omega L}e^{j\left(\psi_u - \frac{\pi}{2}\right)}. \qquad (6.12)$$

That is current in the inductance *L*, according to (6.9) and (6.12), lags in phase from the voltage by the angle φ.

In going from the complex image to the original, we will obtain from (6.9), (6.10)–(6.12)

$$u = U_m\cos(\omega t + \psi_u),$$

$$i_g = gU_m \cos(\omega t + \psi_u),$$

$$i_C = \omega C U_m \cos\left(\omega t + \psi_u + \frac{\pi}{2}\right),$$

$$i_L = \frac{U_m}{\omega L} \cos\left(\omega t + \psi_u - \frac{\pi}{2}\right).$$

In Fig. 6.7 vector diagrams for the r, L, C — circuit in Fig. 1.9, b are shown. Here, in Fig. 6.7, a, the current $\dot{J}_m$ leads the voltage U_m in phase. The angle φ measured from the current to voltage, is negative.

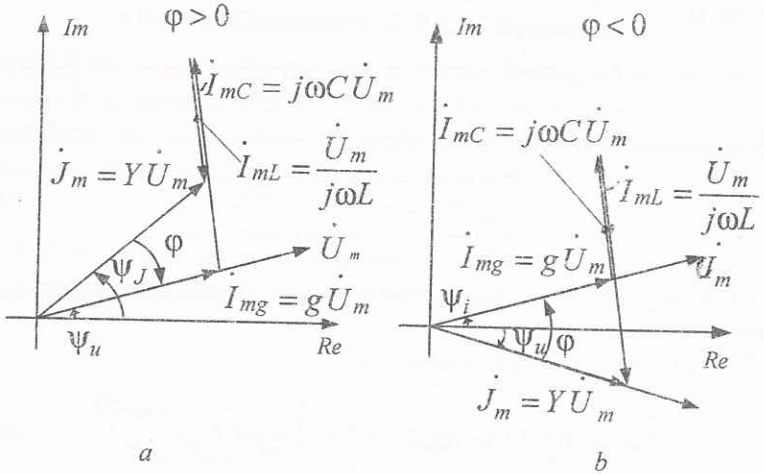


Fig. 6.7

The current $\dot{J}_m$ in Fig. 6.7, b lags in phase from the voltage $\dot{U}_m$. The angle φ is positive.

The circuit as a whole is inductive in nature.

Dividing all values of the vector diagrams in Fig. 6.7 by the voltage $\dot{U}_m$, we get the corresponding vector diagrams for conductances (Fig. 6.8).

Here the angle φ is measured from the admittance Y.

The vector diagrams (Fig. 6.8, a and b) for a capacitive load ($\varphi < 0$) are equivalent.

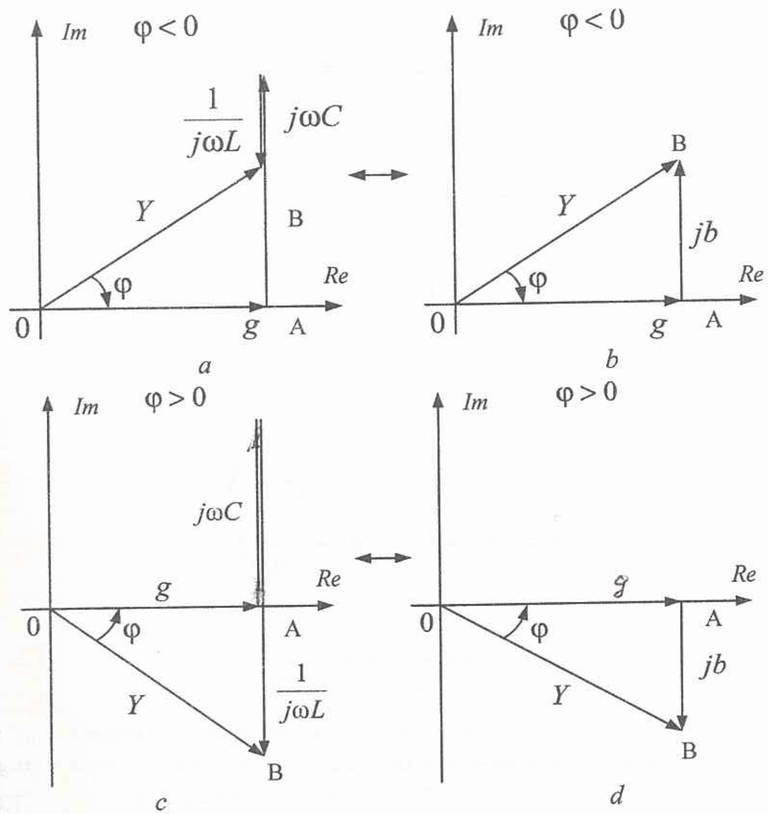


Fig. 6.8

Similarly, the vector diagrams in Fig. 6.8, c and d for an inductive load ($\varphi > 0$) are equivalent. The triangles 0AB in Fig. 6.8 are conductance triangles.

6.5. The Harmonic Current Circuit with a Parallel Connection of *R*, *L* Elements

Relationships for this kind of circuit can be derived from expressions of section 6.4 when $C = 0$. Indeed we get

$$b_C = \omega C = 0.$$

That is, the capacitance C in Fig. 1.9, b can be replaced by a break in the branch with the capacitance. Then from (6.16) we obtain

$$j = i_g + i_L$$

or

$$\mathcal{J}_m \cos(\omega t + \psi_j) = ug + \frac{1}{L}\int_0^t u\,d\tau.$$

From (6.8) we get

$$\dot{\mathcal{J}}_m = \left(g + \frac{1}{j\omega L}\right)\dot{U}_m = Y\dot{U}_m,$$

where

$$Y = g - j\frac{1}{\omega L} = g - jb = ye^{-j\varphi}.$$

Susceptance

$$b = b_L = \frac{1}{\omega L}.$$

Admittance

$$y = \sqrt{g^2 + b^2} = \sqrt{g^2 + \left(\frac{1}{\omega L}\right)^2}.$$

The phase angle

$$\varphi = \text{atan}\frac{b}{g} = \text{atan}\frac{1}{g\omega L}.$$

The voltage at the terminals of the circuit and the currents in the conductance g and inductance L are defined by expressions (6.9), (6.10) and (6.12). The expression for instantaneous values of voltages and currents are identical with expressions obtained in section 6.4. In Fig. 6.9 vector diagrams for the voltage and currents in the circuit (*a*) and conductances (*b*) are given.

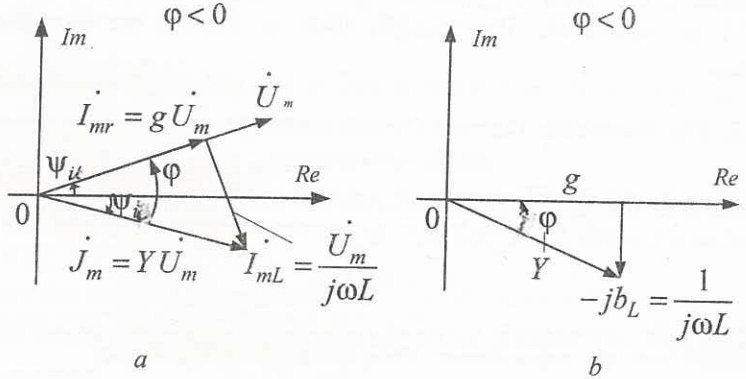


Fig. 6.9

6.6. The Harmonic Current Circuit with a Parallel Connection of *R*, *C* Elements

Relationships for this kind of circuit can be derived from expressions of section 6.4 when $L \to \infty$. Indeed we get

$$b_L = \frac{1}{\omega L} = 0.$$

That is the inductance L in Fig. 1.9, *b*, can be replaced by a break in the branch with the inductance. Then

$$j = i_g + i_C$$

or

$$\mathcal{J}_m \cos(\omega t + \psi_j) = ug + C\frac{du}{dt}.$$

From (6.8) we obtain

$$\dot{\mathcal{J}}_m = (g + j\omega C)\dot{U}_m = Y\dot{U}_m,$$

where

$$Y = g + j\omega C = g - jb = ye^{-j\varphi}.$$

Susceptance

$$b = -b_C = -\omega C.$$

Admittance

$$y = \sqrt{g^2 + b^2} = \sqrt{g^2 + (\omega C)^2}.$$

The phase angle

$$\varphi = \text{arctg}\frac{b}{g} = \text{atan}\left(-\frac{\omega C}{g}\right) = -\text{atan}\frac{\omega C}{g}.$$

The voltage across the terminals of the circuit and the currents in the conductance g and capacitance C are defined by expressions (6.10) and (6.11). The expressions for instantaneous values of voltages and currents are identical to those in section 6.4. Fig. 6.10 shows vector diagrams for the voltages and currents in the circuit (*a*) and conductances (*b*).

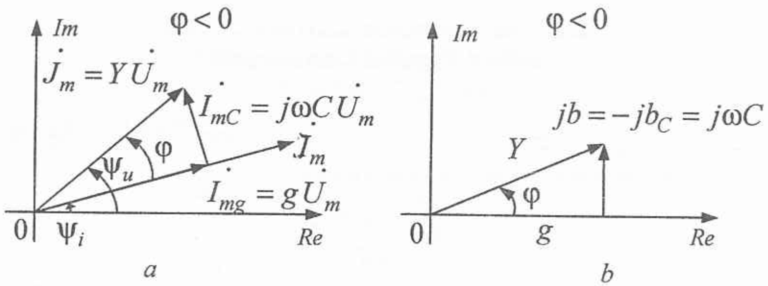


Fig. 6.10

Example 1

Calculate the frequency at which the reactive component of the input complex impedance of the circuit (Fig. 6.11) is equal to zero.

The parameters of the circuit: $L = 0{,}1$ H; $C = 0{,}2$ mcF; $R = 2$k0 hms.

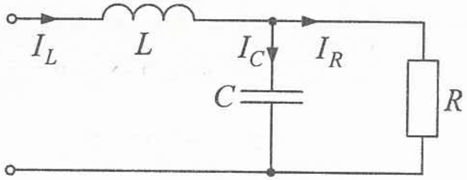


Fig. 6.11

Solution

Find the complex impedance of the circuit. The complex admittance of a parallel-connected resistance and a capacitance is

$$Y_{RC} = Y_R + Y_C = \frac{1}{R} + j\omega C.$$

The complex input impedance

$$Z_{inp} = Z_L + \frac{1}{Y_{RC}} = j\omega L + \frac{1}{\frac{1}{R} + j\omega C} = j\omega L + \frac{R}{1 + j\omega C} =$$

$$= j\omega L + \frac{R(1 - j\omega RC)}{(1 + j\omega RC)(1 - j\omega RC)} = jvL + \frac{R - jvR^2C}{1 + (\omega RC)^2} =$$

$$= \frac{R}{1 + (\omega RC)^2} + j\left[\omega L - \frac{vR^2C}{1 + (\omega RC)^2}\right].$$

The reactive component of the complex input impedance

$$X_{inp} = \omega L - \frac{\omega R^2 C}{1 + (\omega RC)^2}.$$

For $X_{inp} = 0$, we obtain

$$\omega L - \frac{\omega R^2 C}{1 - (\omega RC)^2} = 0; \; L - \frac{R^2 C}{1 + (\omega RC)^2} = 0.$$

Hence

$$\omega = \sqrt{\frac{1}{LC} - \frac{1}{R^2C^2}} = \sqrt{\frac{1}{0{,}1 \cdot 0{,}2 \cdot 10^{-6}} - \frac{1}{(2 \cdot 10^3)^2 (0{,}2 \cdot 10^{-6})^2}} = 6{,}6 \cdot 10^3 \text{s}^{-1}.$$